Non-Euclidean geometry and Special Relativity

Seppe

Photo from WSU

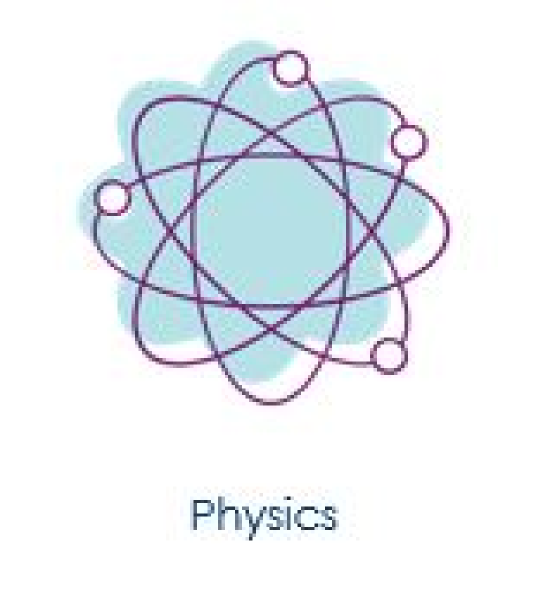
Fact file

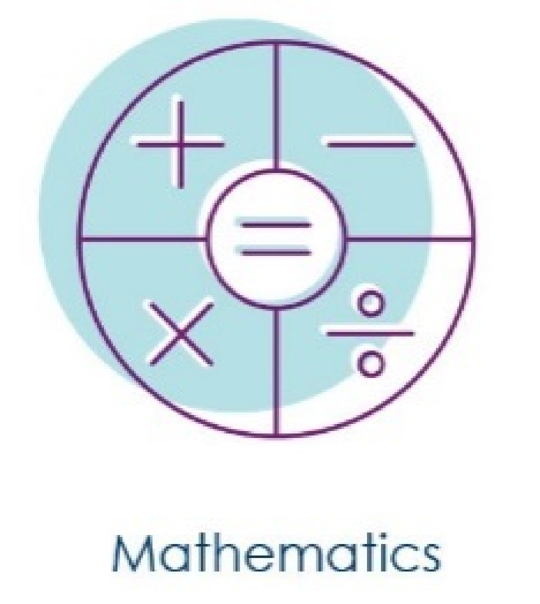
- Seppe (he/him)
- St. Edmund’s College
- Department of Applied Math. And Theoretical Phys. (DAMTP)
- Black Holes, Gravitational Waves, General Relativity
- PhD: Numerical Relativity and Gravitational Waves
- Belgium, Leuven

My Academic Pathway

- Greek-Mathematics, SALCO Haasrode, BE
- TWIN Bachelor Math.-Phys., KU Leuven, BE (2020)
- MSc Theoretical Phys. (2022), MSc Astrophysics (2023), KU Leuven, BE
- PhD (2023 – 2027), Cambridge.
- Why PhD? Passion for the Universe, teaching, academic lifestyle.
- Afterwards: Postdoc? Professorship ?

- Euclidean geometry
- Riemannian geometry
- Special Relativity
- General Relativity





Overview

Learning Objectives

- Discover some concepts of non-Euclidean geometry.
- Have a basic understanding of the mathematical principles governing Special / General Relativity.

Reminder of Pre-work

Do

- Look up the fastest route for a plane to travel between London and New York City, and draw it on a flat world map. Why is this not a straight line?

Watch

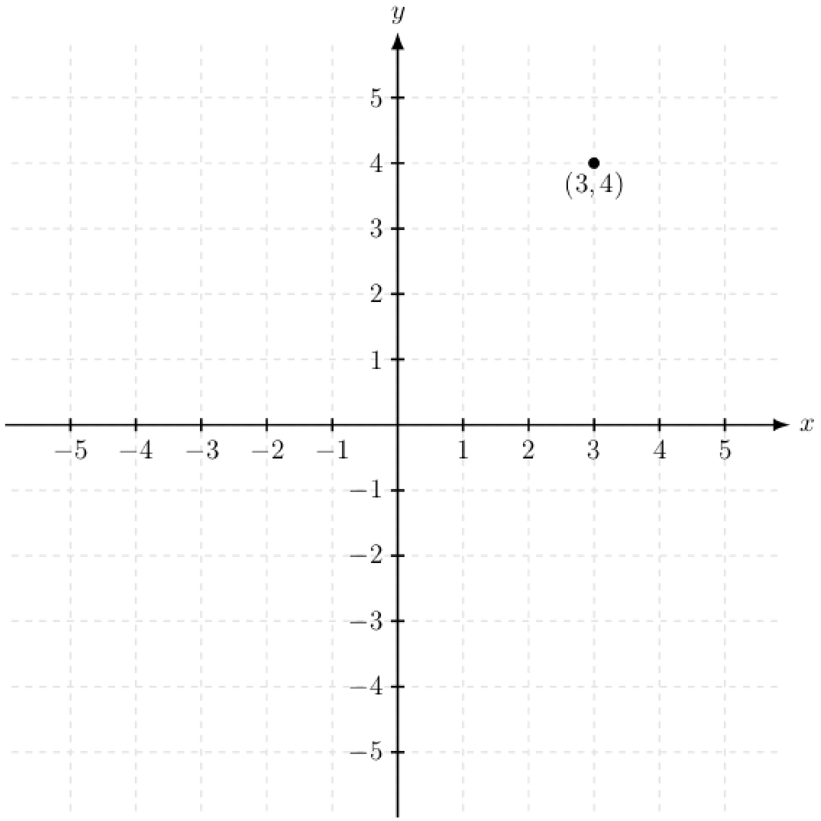
- Watch the following Veritasium video until 19:58
  - https://youtu.be/lFlu60qs7_4?si=wTdhMoXerqpfGbQV
- Can you now explain why the shortest path between London and New York does not seem straight on a flat world map? Does it look straight on the Earth’s sphere?

Non-Euclidean geometry

Euclidean plane

- 2-dimensional vectors $(x, y) \in \mathbb{R}^2$
- Euclidean distance

$$\mathbf{a} = (x_1, y_1)\,, \mathbf{b} = (x_2, y_2)$$
$$d(\mathbf{a}, \mathbf{b}) = D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$D^2 = \Delta x^2 + \Delta y^2$$



From [1]

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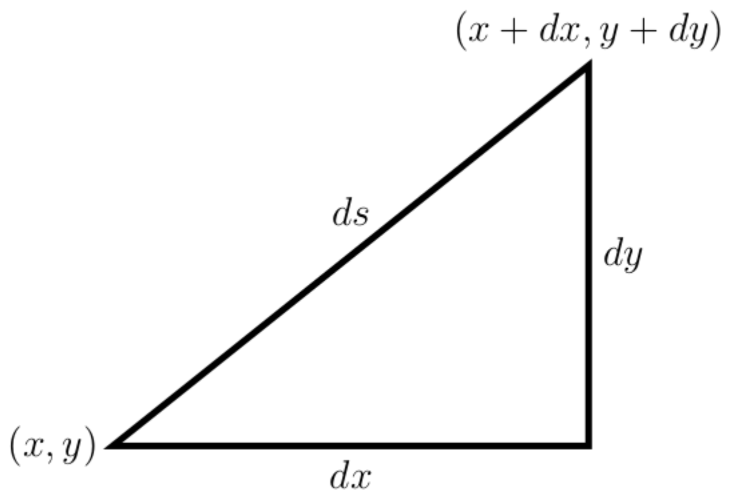
- For small distances

$$ds^2 = dx^2 + dy^2$$



From [1]

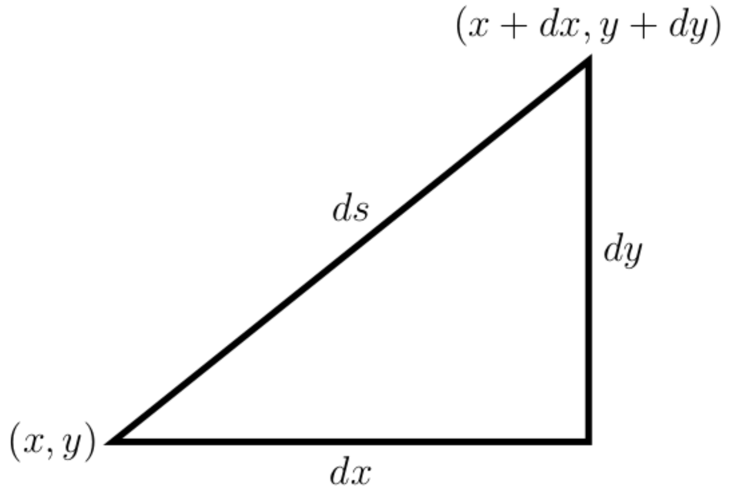
N-dimensional Euclidean space

- N-dimensional vectors $(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$
- Euclidean distance

$$ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_N^2$$
$$= \sum_{i=1}^{N} dx_i^2$$

- Recall:

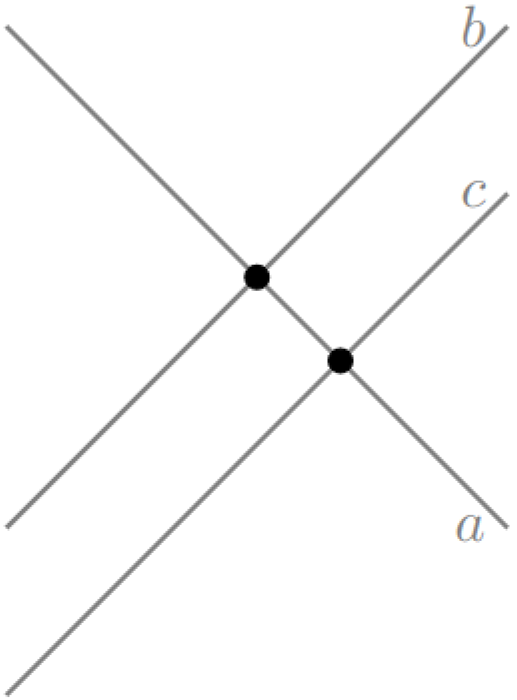
$$ds^2 = dx^2 + dy^2$$



From [1]

Euclidean geometry

- Through any 2 different points goes exactly one straight line.
- Two parallel lines either are the same, or never intersect.
- Two different straight lines intersect at most once.

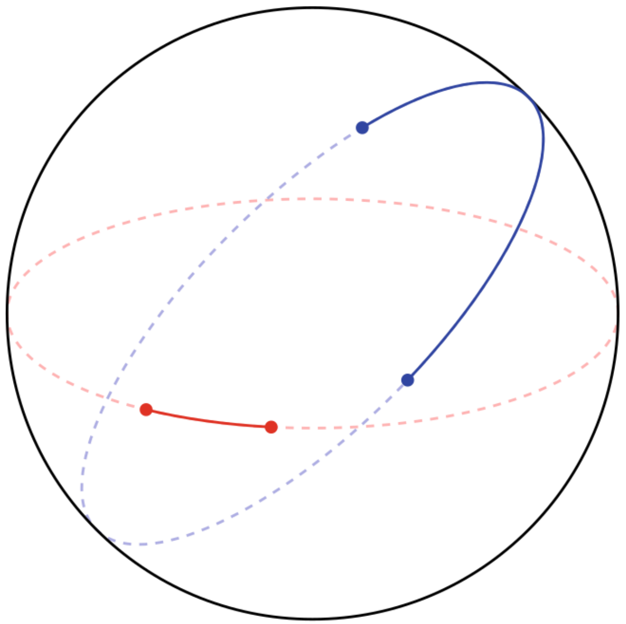


Geometry on a sphere

- Locally (!), our Earth looks flat. Euclidean geometry.
- However, the Earth is a sphere (it really is!)
- How long are straight lines on the Earth’s surface?
- What even is a straight line on Earth’s surface?

Geometry on a sphere

From [1]

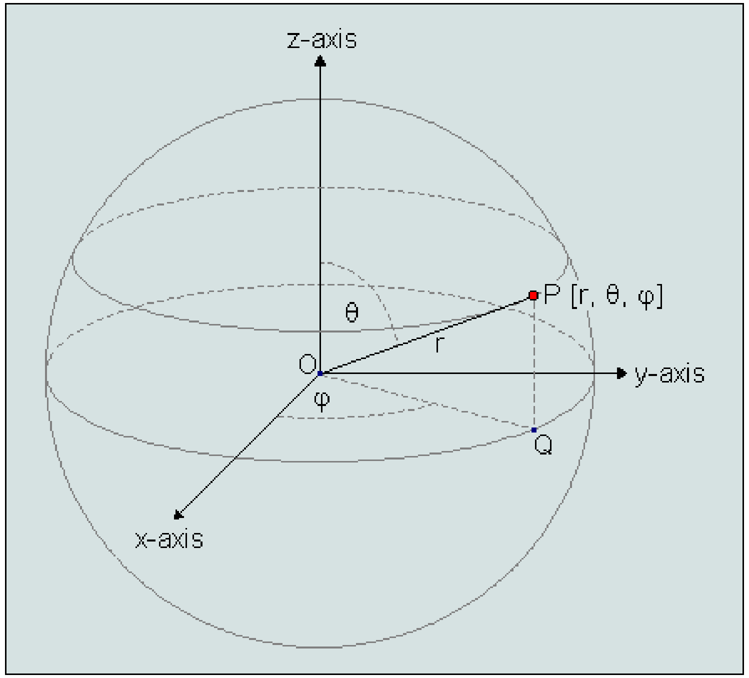


- Straight lines: the shortest possible path between two points.
- This is called a *geodesic*.
- On a sphere: great circles
- Great circles (when extended) intersect twice
- No “parallel straight lines” exist

The metric

- Points on a sphere with radius R $(\theta, \phi) \in S^2$
- Distance between two nearby points

$$ds^2 = R^2 \left(d\theta^2 + \sin^2 \theta \, d\phi^2 \right)$$



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- Distance between two nearby points

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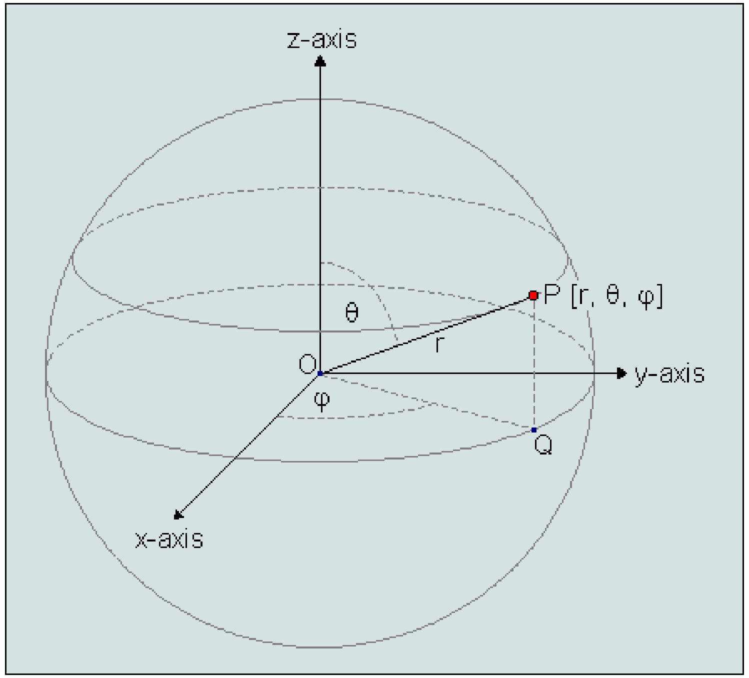
- Distance between two distant points?

$$(\theta_1, \phi_1), (\theta_2, \phi_2)$$

$$\int_{(\theta_1,\phi_1)}^{(\theta_2,\phi_2)} ds = \int_{(\theta_1,\phi_1)}^{(\theta_2,\phi_2)} R\sqrt{d\theta^2 + \sin^2\theta \, d\phi^2}$$



The metric

- Example: sphere of radius 1

$$(0,0,1) \to (0,1,0) \in \mathbb{R}^3$$

$$(0,0) \to \left(\frac{\pi}{2}, 0\right) \in S^2$$

- Euclidean distance:

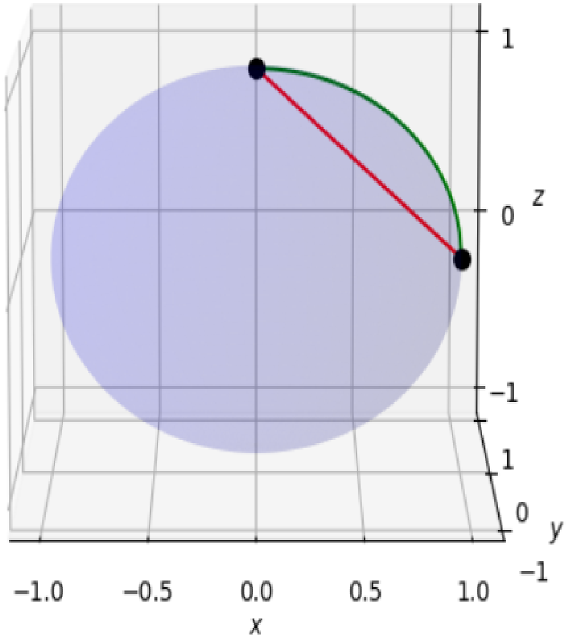
$$ds^2 = dx^2 + dy^2 + dz^2$$

$$D = \sqrt{2} = 1.41\ldots$$

- Distance on the sphere

$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

$$D = \frac{\pi}{2} = 1.57\ldots$$



Riemannian geometry

- Clearly, distances depend on the metric used.
- Riemann: generalization of everything before
- General metrics

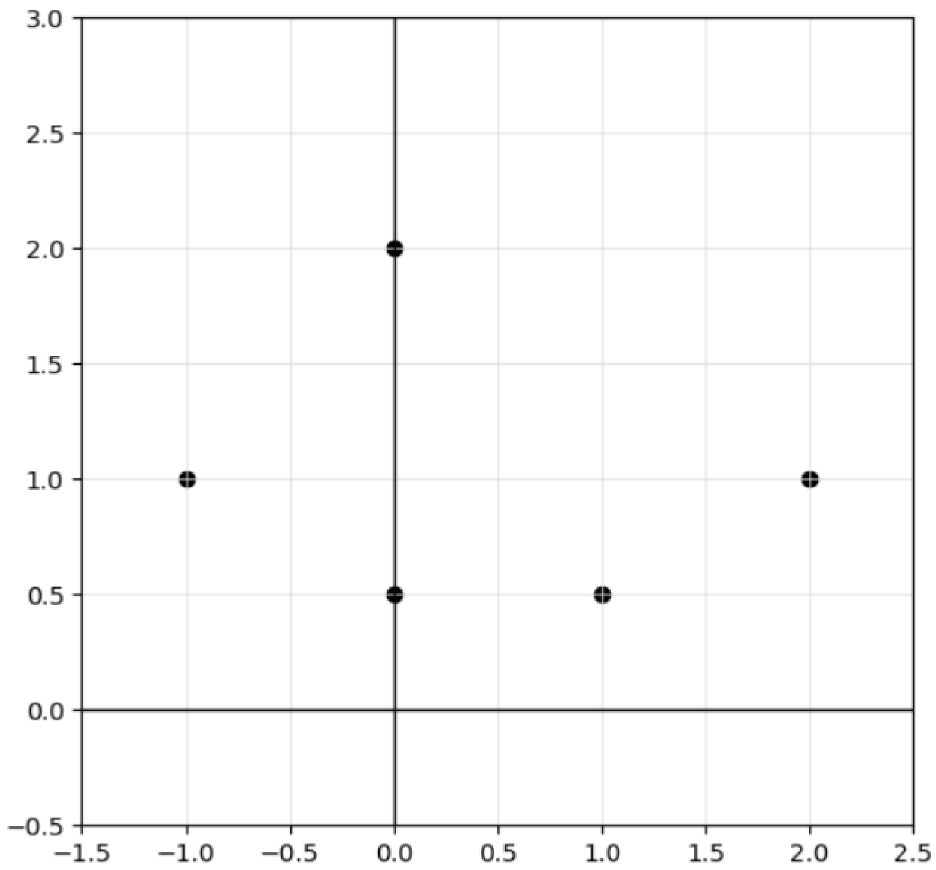
$$ds^2 = g_{11}(\mathbf{x})dx_1^2 + g_{22}(\mathbf{x})dx_2^2 + g_{12}(\mathbf{x})dx_1 dx_2 + \ldots$$

- Recall: $ds^2 = R^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$
- Smooth manifolds: sphere, torus, …
- Importantly: ds^2 is always positive.

The Poincaré half-plane

- A famous example in 2 dimensions.
- Upper half of real plane $\{(x, y) \in \mathbb{R}^2 | \, y > 0\}$
- Metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$



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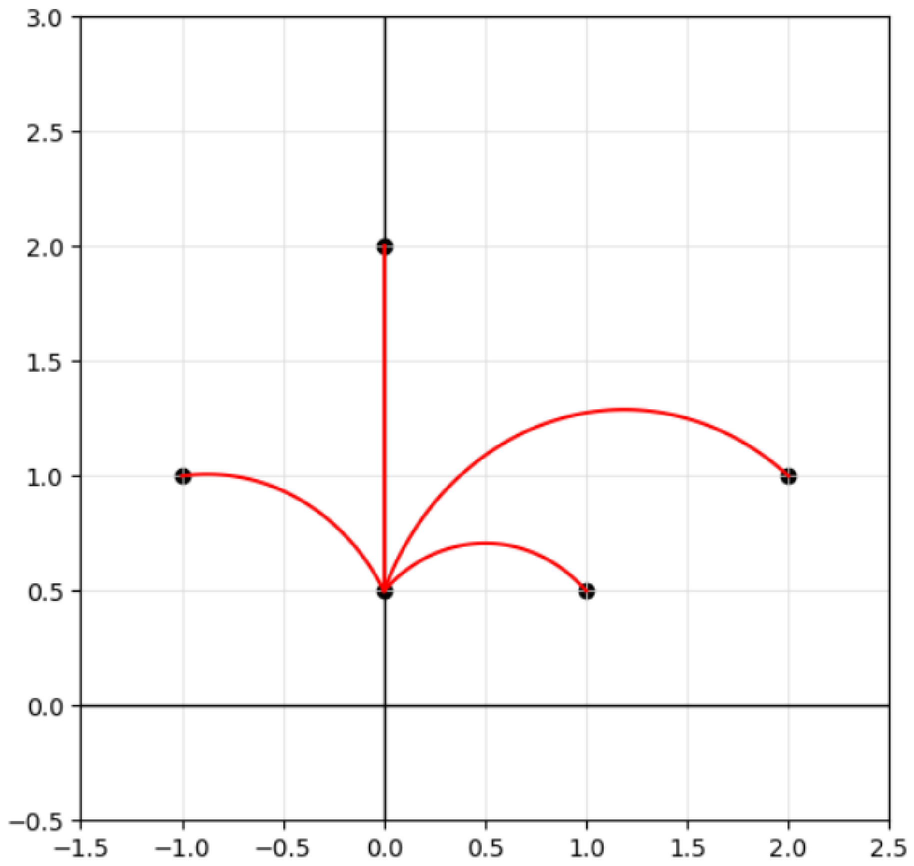
- Metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

- Straight lines (i.e. minimizing length between two points) are
  - Arcs of circles with centre on x-axis
  - Straight vertical lines



Special Relativity

Relativity

- At the start of the 20th century, physicists had constructed a “classical” world view (no quantum mechanics or special relativity). Physics seemed to be “finished”.
- Principle of equivalence: different “inertial observers” must see the same physics.
- Different reference frames are related by “Galilean transformations”

$$\begin{cases} t' = t, \\ x' = x - ut, \\ y' = y, \\ z' = z. \end{cases}$$

$$\frac{\Delta x'}{\Delta t} = \frac{\Delta x}{\Delta t} - u$$

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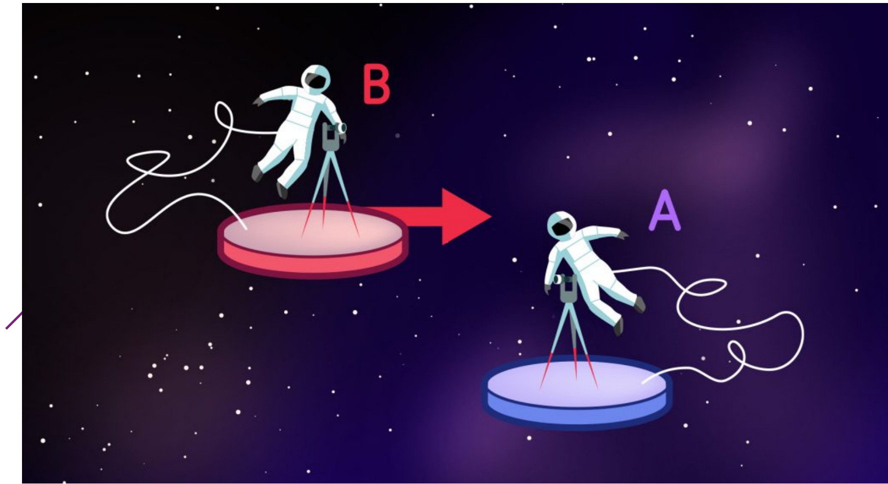

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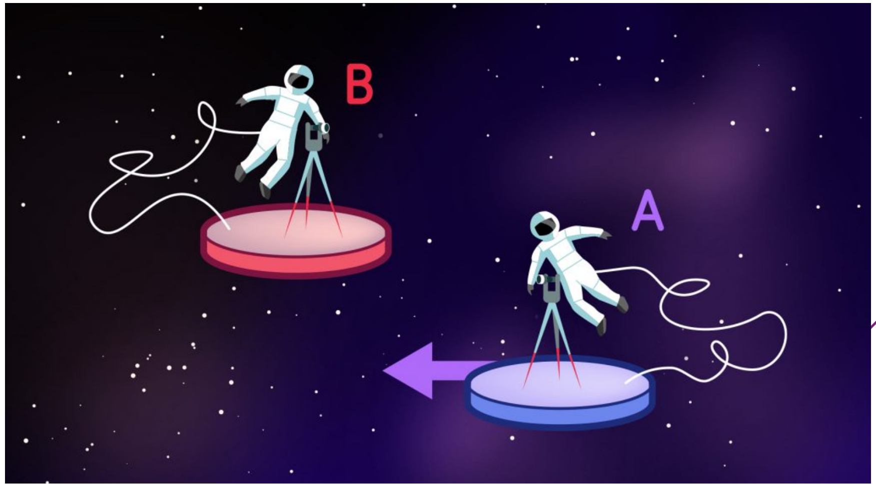

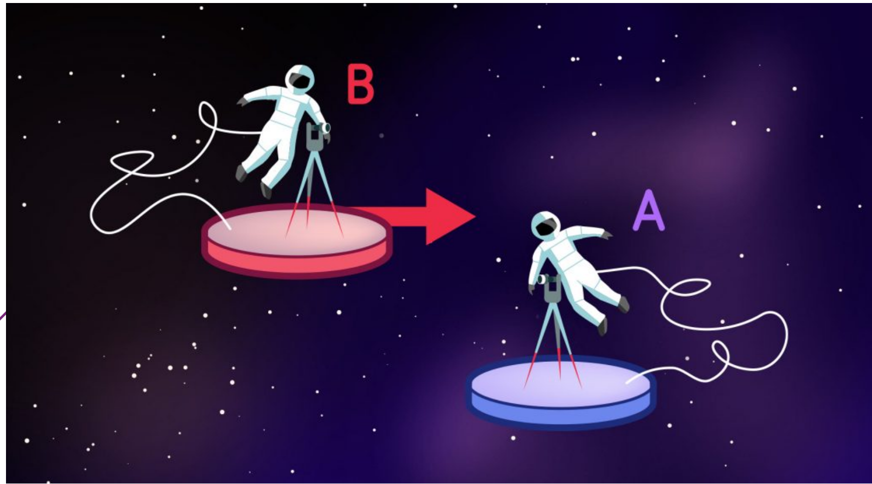

Special Relativity

- However, the laws of electromagnetism are inconsistent with all of this
- Solution: the speed of light must be different for different observers
- Experiments (like Michelson-Morley) showed however that <u>the speed of light is the same for all observers!</u>

Special Relativity

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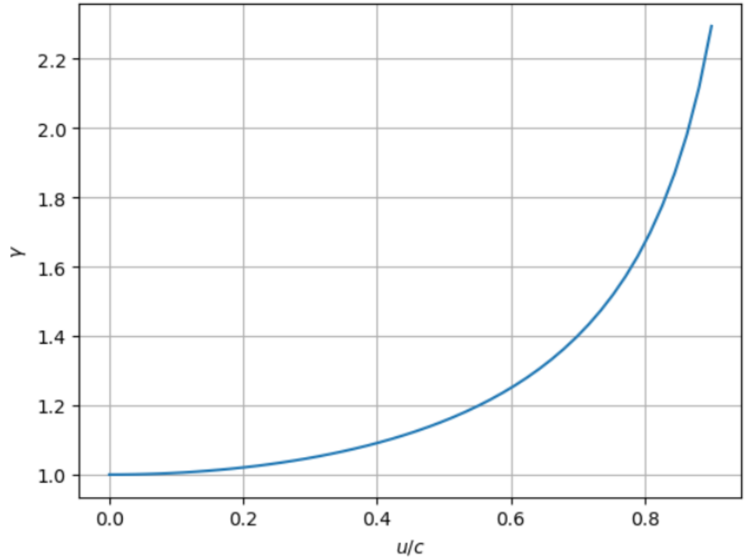
- Solution: the speed of light must be different for different observers
- Experiments (like Michelson-Morley) showed however that <u>the speed of light is the same for all observers!</u>
- Einstein’s solution: the Galilean transformations are not correct!
- Lorentz transformations



$$\begin{cases} t' = t, \\ x' = x - ut, \\ y' = y, \\ z' = z. \end{cases}$$

$$\begin{cases} t' = \gamma\left(t - \frac{u}{c^2}x\right), \\ x' = \gamma\,(x - ut), \\ y' = y, \\ z' = z. \end{cases}$$

$$\gamma = \sqrt{\frac{1}{1 - \frac{u^2}{c^2}}}$$

Length contraction

- Observer B has a ruler of length 1m:

$$\Delta x' = x'_1 - x'_0 = 1\text{m}$$

- From the other observer's (A) perspective:

$$\begin{cases} t' &= \gamma\left(t - \frac{u}{c^2}x\right), \\ x' &= \gamma(x - ut) \end{cases}$$

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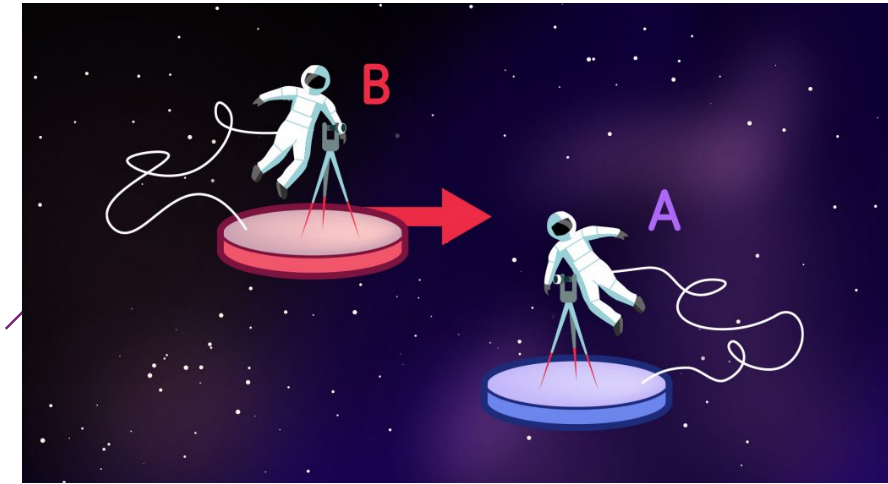

Length contraction

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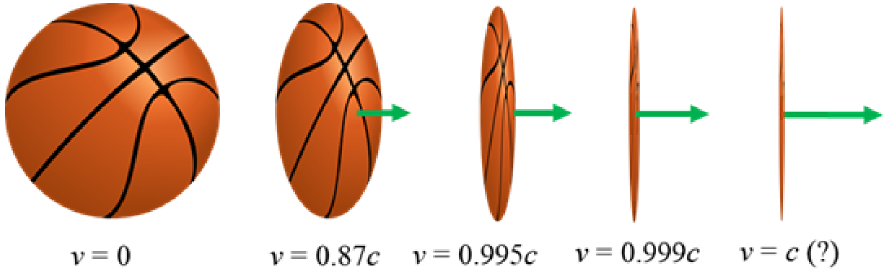
$$\Delta x' = x_1' - x_0' = 1\text{m}$$

- From the other observer's (A) perspective:

$$\Delta x' = \gamma(x_1 - x_0 - u(t_1 - t_0)) = \gamma \Delta x$$

$$\Delta x = \frac{1\text{m}}{\gamma}$$

$$\begin{cases} t' & = \gamma\left(t - \frac{u}{c^2}x\right), \\ x' & = \gamma(x - ut) \end{cases}$$



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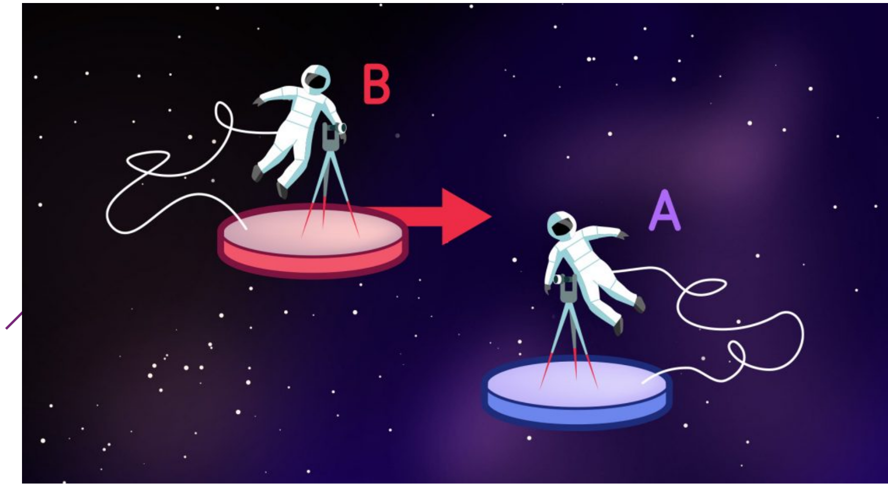
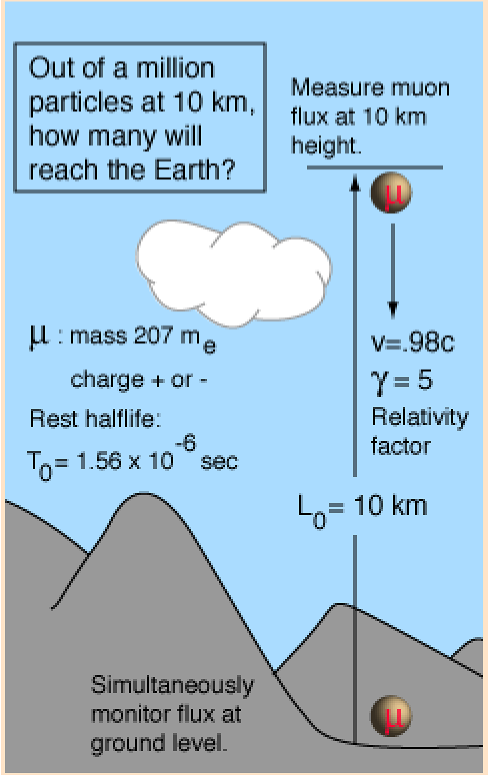


Time dilation

- Similarly, one can derive $$\Delta t = \gamma \Delta t'$$
- Time experienced by a moving observer goes slower.
- Example: muon decay

Source: Link



In the muon experiment, the relativistic approach yields agreement with experiment and is greatly different from the non-relativistic result. Note that the muon and ground frames do not agree on the distance and time, but they agree on the final result. One observer sees time dilation, the other sees length contraction, but neither sees both.

	Relativistic		Non-Relativistic
	Muon	Ground	
Distance	2 km	10 km	10 km
Time	6.8µs	34µs	34µs
Halflives	4.36	4.36	21.8
Surviving	49000	49000	0.3

These calculated results are consistent with historical experiments.

Minkowski metric

- If the speed of light is the same for everyone:

$$\frac{\Delta x}{\Delta t} = c \quad \Leftrightarrow \quad 0 = -c^2 \Delta t^2 + \Delta x^2$$

Minkowski metric

- If the speed of light is the same for everyone:

$$\frac{\Delta x}{\Delta t} = c \quad \Leftrightarrow \quad 0 = -c^2 \Delta t^2 + \Delta x^2$$

- Einstein introduced a metric on a 4-dimensional “space”: *spacetime*

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

- Invariant under Lorentz transformations
- Lorentz transformations and Minkowski spacetime satisfy the principle of relativity and a constant speed of light.

Minkowski metric

- All of this is confusing
- Minus sign in the metric?
- 4-dimensional spacetime?
- Lorentz transformations?

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

Minkowski metric

- All of this is confusing
- Minus sign in the metric?
- 4-dimensional spacetime?
- Lorentz transformations?
- Simplify
- One space, one time dimension

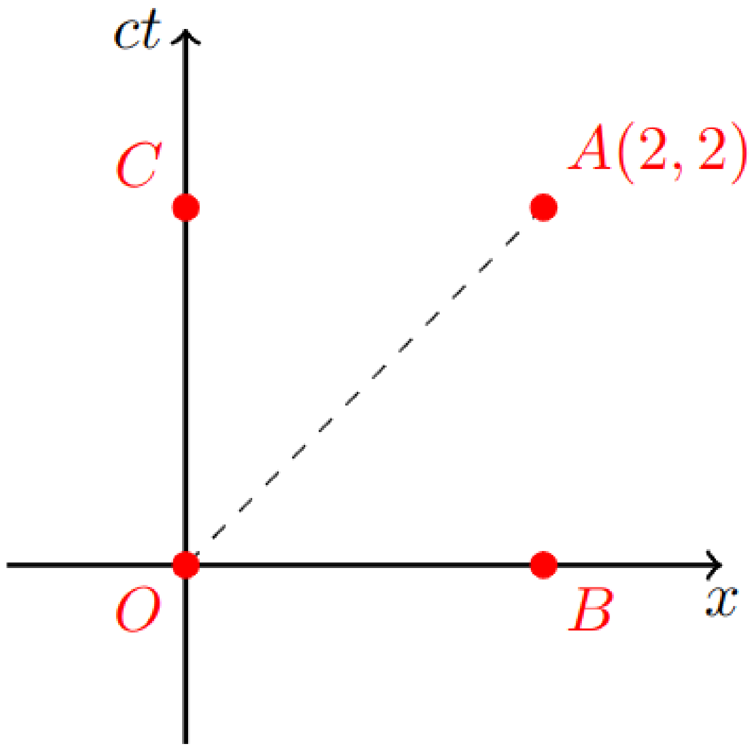
$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

$$ds^2 = -c^2dt^2 + dx^2$$

Lengths of curves

$$ds^2 = -c^2dt^2 + dx^2$$

- What does this minus sign in the metric do?
- Length of OB: as usual, equal to 2



Lengths of curves

$$ds^2 = -c^2 dt^2 + dx^2$$

- What does this minus sign in the metric do?
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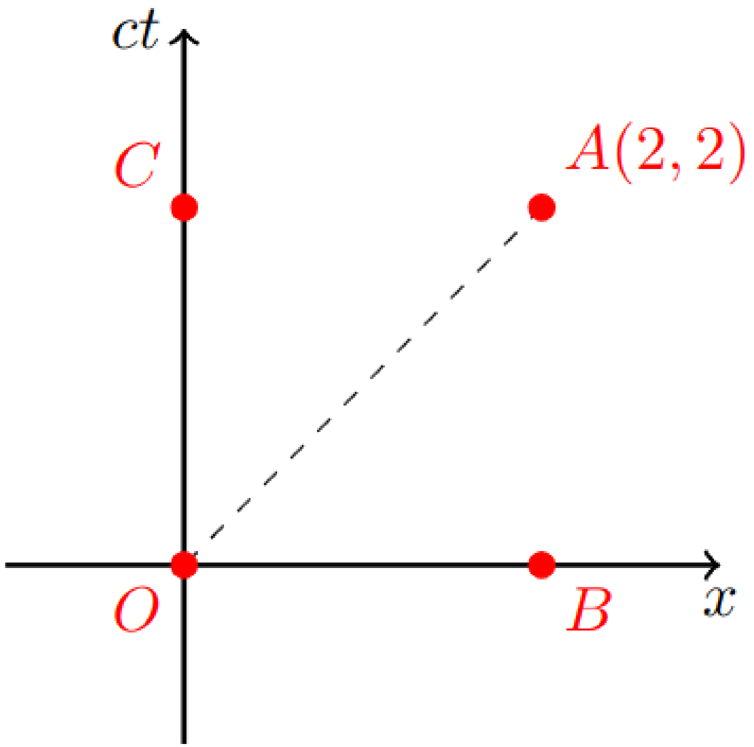
- Length of OC: no change in x, only in time.

$$ds^2 = -c^2 dt^2 \qquad \Delta s = -\Delta(ct) = -2 \text{ ?}$$



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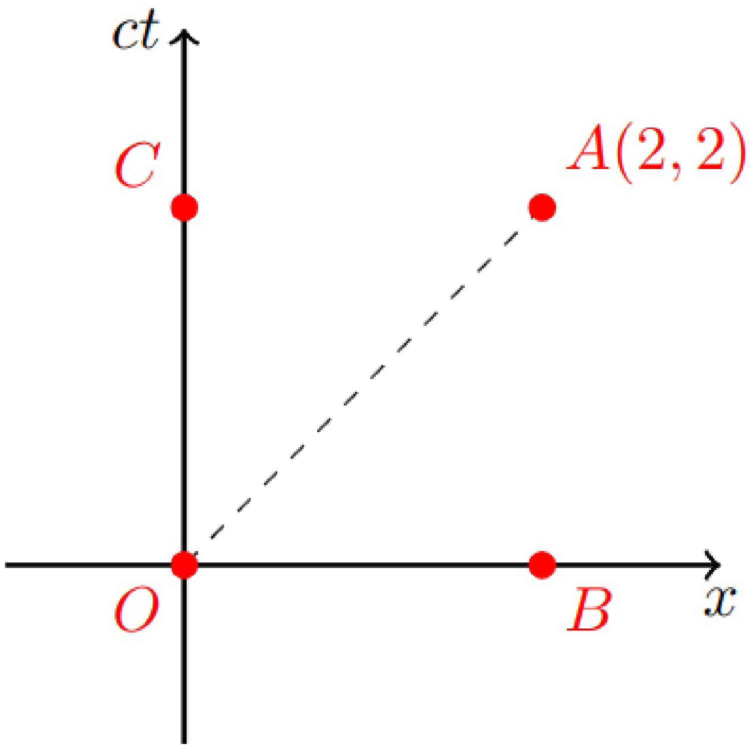
(this is not exactly correct)

- Length of OA: change in x is the same as the change in ct.

$$\Delta s = -\Delta(ct) + \Delta x = 0$$



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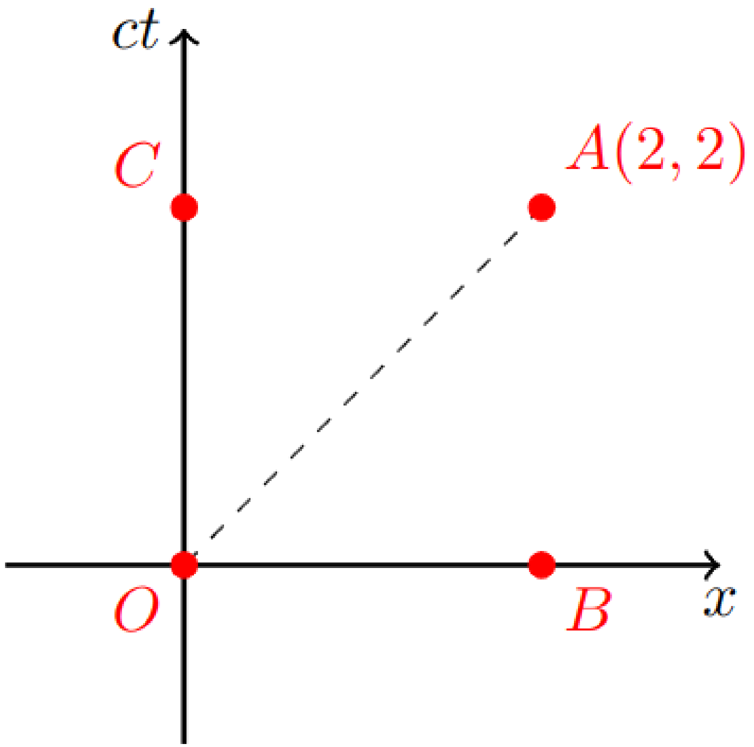
- In Minkowski space: curves can also have zero length.



Lengths of curves

$$ds^2 = -c^2 dt^2 + dx^2$$

- What does this minus sign in the metric do?
- Length of OB: as usual, equal to 2. Spacelike separated
- Length of OC: no change in x, only in time. Timelike

$$ds^2 = -c^2 dt^2 \qquad \Delta s = -\Delta(ct) = -2 \text{ ?}$$

(this is not exactly correct)

- Length of OA: change in x is the same as the change in ct.

Lightlike $\Delta s = -\Delta(ct) + \Delta x = 0$

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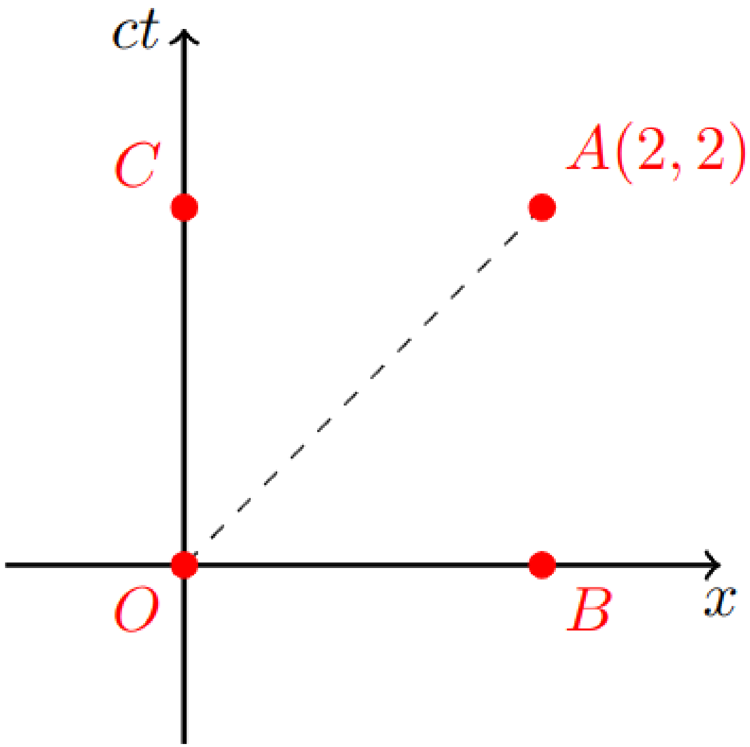
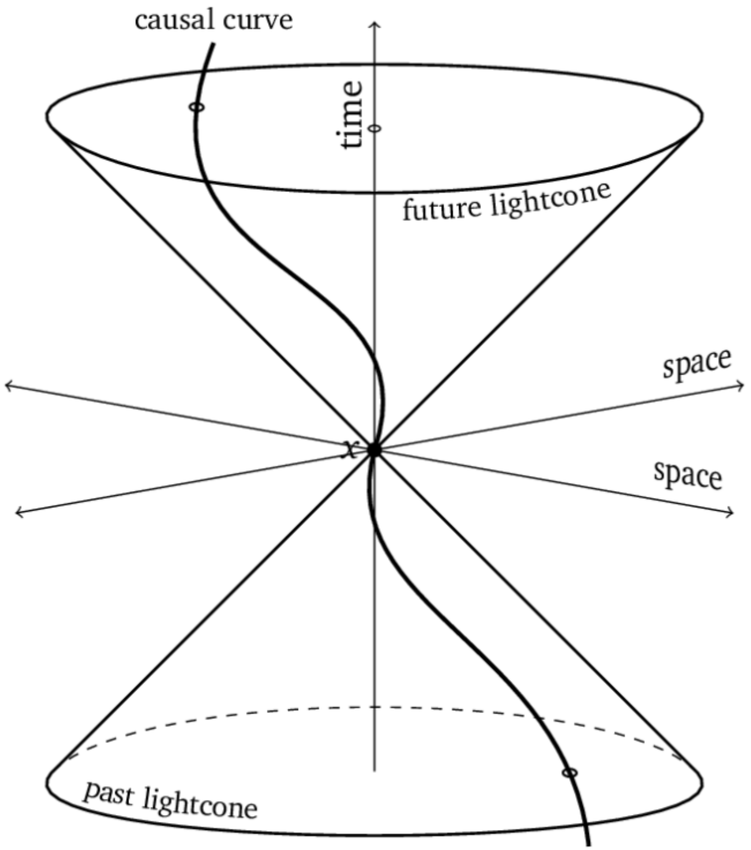


Minkowski spacetime

(Norton, 2015)

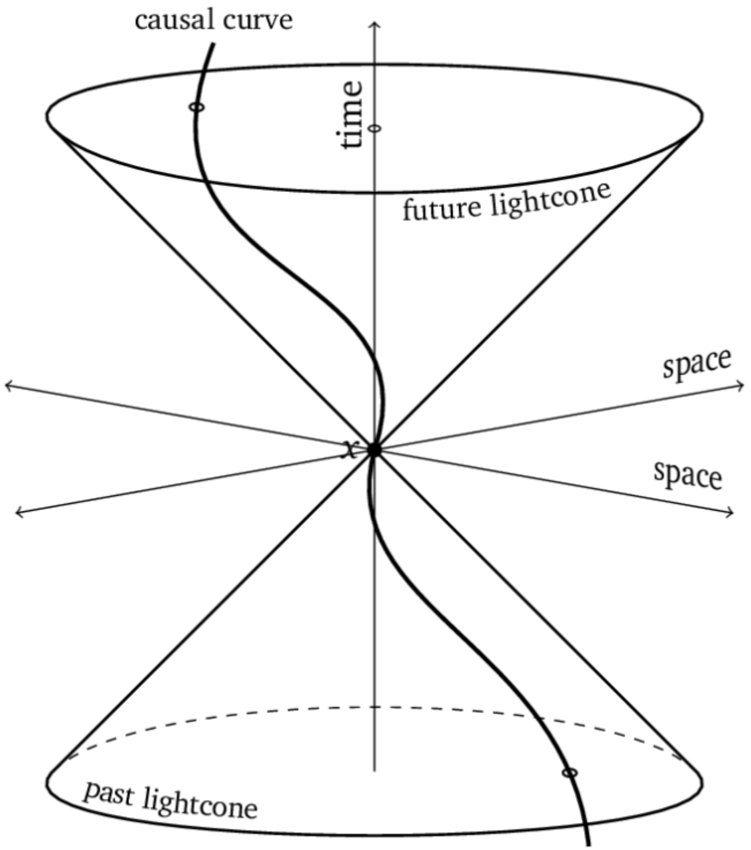


- The light cone at a point are all the points in spacetime that are in *causal contact* with that point

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

Special Relativity

(Norton, 2015)



- To obey the principle of relativity, with every inertial observer measuring the same speed of light, Einstein introduced Lorentz transformations.
- The Lorentz transformations leave the spacetime interval (the metric) invariant

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

- The minus sign in the metric indicates a timelike direction.
- The distance between two different points can be zero now as well.
- The lightcone defines the points in spacetime that are in causal contact

General Relativity

Theory

- Equality of gravitational and inertial mass

$$F_{grav} = m_{grav}\, g \qquad\qquad F = m_{inert}\, a$$

$$a = \frac{m_{grav}}{m_{inert}}\, g = k\, g$$

- There is no law that prohibits k from differing between different particles / matter.
- But, the masses are the same

$$a = g$$

Theory

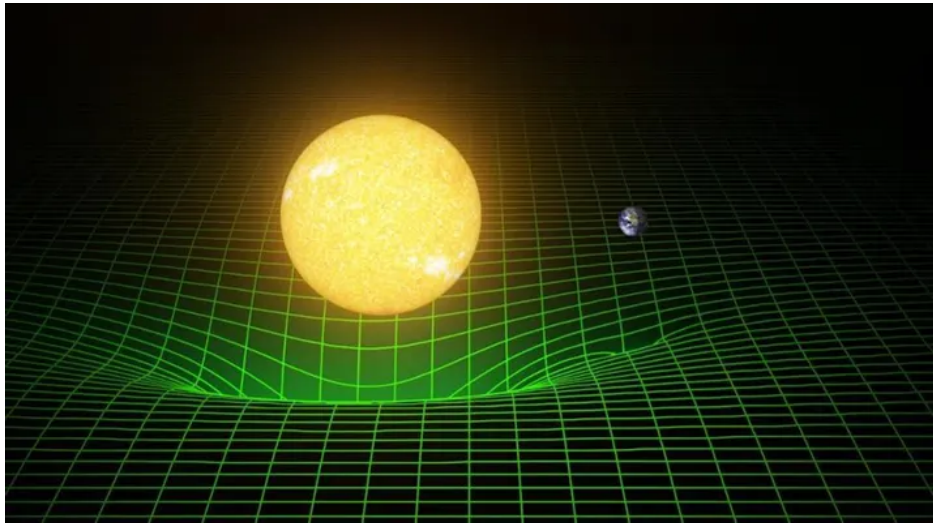
- Einstein: locally, there is no way to distinguish between an observer at rest and an observer falling in a uniform gravitational field.
- Einstein's equivalence principle: All observers are equivalent if there are no forces than the gravitational force working on the observer.
- This puts the gravitational force on a special pedestal.

Theory

- Einstein: locally, there is no way to distinguish between an observer at rest and an observer falling in a uniform gravitational field.
- Einstein's equivalence principle: All observers are equivalent if there are no forces than the gravitational force working on the observer.
- This puts the gravitational force on a special pedestal.
- Einstein: gravity is the result of *curvature* in spacetime.
- Spacetime is still described with a metric.
- **Curvature <=> Matter / Energy**

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

T. PYLE/CALTECH/MIT/LIGO LAB



Geodesics

- How to determine the motion of masses in curved spacetime?
- Geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0$$

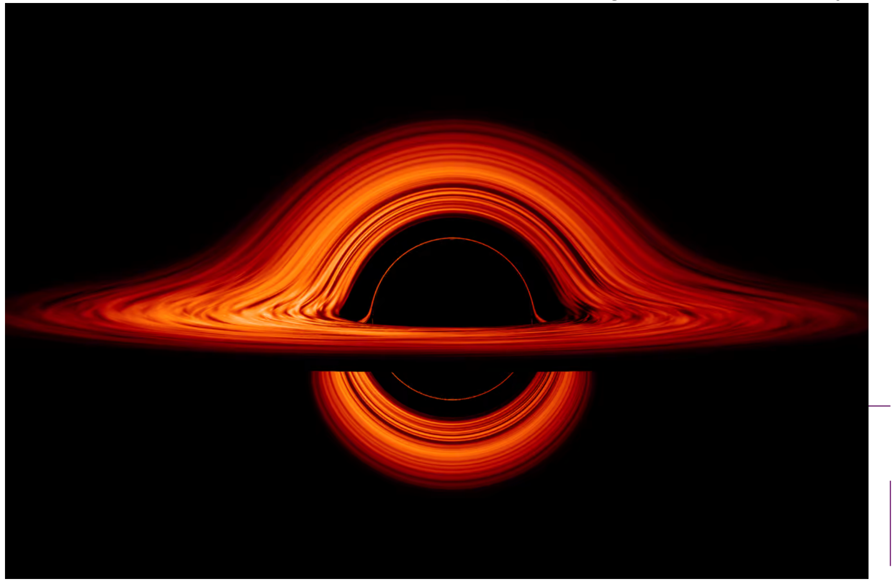
- Geodesics: paths that extremise the space-time distance between two points.
- The curvature of spacetime tells particles how to move. Particles upon which no forces *other than gravity* are acting, follow geodesics.

Black holes

- Black holes are solutions to Einstein's equations for vacuum
- Schwarzschild black hole (non-rotating) metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)(c^2 dt^2) + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

NASA's Goddard Space Flight Center/Jeremy



- Schwarzschild radius r = 2M: event horizon
- From the metric, all the science can be derived.

Conclusions

Takeaways

- In geometry, we can use metrics that differ from the Euclidean distance.
- When we use different metrics, the notion of a ‘straight line’ is replaced with the notion of a geodesic; a line that minimises the length between two points.
- Special Relativity is described in terms of the Minkowski metric on a 4-dimensional spacetime, where the time coordinate has a minus sign in the metric.
- The introduction of the minus sign leads to the possibility of zero lengths.
- In General Relativity, more complex metrics describe different physical situations. The time component still has a minus sign, and geodesics are obtained from the geodesic equation.

Suggestions for Further Research

Read

- Reference [1] is an excellent presentation of geometry, Special and General Relativity aimed at students finishing high school. This is not free, however.

Watch

- PBS Spacetime (YouTube) has multiple excellent videos on Relativity, with stunning animations. Playlist on General Relativity: https://youtube.com/playlist?list=PLsPUh22kYmNAmjsHke4pd8S9z6m_hVRur&si=8vdTEMYj1DMIqEY_
- CCIR Future Scholar Programme: normally offers a 12 week online course on Special Relativity

Extension Questions

- Try to show that the Minkowski metric is invariant under Lorentz transformations. For this, consider the Lorentz transformation as shown in the slides, for a velocity that is purely in the x-direction. To calculate dx’ and dt’ you need to use the chain rule, e.g.

$$dx = \frac{\partial x}{\partial x'}dx' + \frac{\partial x}{\partial t'}dt'$$

- A Friedmann-Lemaître-Robertson-Walker (FLRW) metric is a model for a homogeneous (the same everywhere in space) and isotropic (the same in all directions) Universe. Taking coordinates t,x,y,z, can you find out what the metric should look like? On what coordinates can the metric depend? How many different metric functions can you have?

- Proper time T is defined as below, and corresponds to the time experienced by an observer in its own rest frame. In other words, proper time is the physical time an observer experiences, rather than the coordinate time t. In terms of the Minkowski metric, can you explain why this definition makes sense?

$$d\tau^2 = -\frac{ds^2}{c^2}$$

Extension Questions: Answers

- $$dx = \gamma(dx' - u\,dt')\,,\, dt = \gamma(dt' - \frac{u}{c^2}\,dx')$$
$$c^2dt^2 - dx^2 = \gamma^2\left(c^2(dt')^2 - (dx')^2 + \frac{u^2}{c^2}(dx')^2 - u^2(dt')^2\right)$$
$$= c^2(dt')^2 - (dx')^2$$

- Due to homogeneity, the metric functions can only depend on time (i.e. the metric does not depend on the spatial position. Due to isotropy, no cross terms dx dy can be in the metric, and all spatial metric functions need to be the same. This leads to the FLRW metric

- An observer in its own rest frame will always be at the origin of its coordinates, and does not measure its own movement (as the frame moves with the observer). Therefore, according to the observer dx=dy=dz=0, and the metric in the observer's frame is simply
$$ds^2 = -c^2dt^2$$
  - Therefore, the proper time is the time coordinate an observer assigns to its own rest frame, i.e. the time coordinate the observer itself experiences (therefore, *proper* time)

Glossary of Key Terms

Key Term	Definition
Metric	The metric determines the distance between neighbouring points. It can vary from point to point (Riemannian) and even be negative (Lorentzian).
Geodesic	A line between two points that extremises (i.e. minimises or maximises) the distance between these points. A generalisation of 'straight lines' in more complicated geometries.
Special Relativity	Einstein's Theory of Special Relativity relates measurements between different inertial observers, that is observers without acceleration, moving at constant velocity with respect to each other. It unifies space and time into 4-dimensional spacetime.
General Relativity	Einstein's Theory of General Relativity extends Special Relativity to include gravity. It describes gravity as a result of the curvature of spacetime, caused by massive bodies.

References

- [1] Mayerson, Daniel R., Anthony M. Charles, and Joseph E. Golec. Relativity: a journey through warped space and time. Springer, 2019.
- [2] Norton, J. (2015, February 9). Spacetime. Einstein for Everyone.
- [3] Carroll, Sean M. Spacetime and geometry. Cambridge University Press, 2019.

Studying Phys/Math at university

- In general: challenging, but very interesting.
- Skills: analytical thinking, (abstract) reasoning, problem solving, perseverance
- Not a study that you undertake for lack of better alternatives: motivation / passion required.
- Physics requires mathematics
- Why? Understanding the Universe, learning an analytical/mathematical way of thinking. The latter is heavily sought after in companies.

A Levels

AAB - ABB

Scottish Highers

AAAAB

Vocational

BTEC DDD

Other Level 3/Level 6 qualifications (e.g. Pearson BTEC Level 3 National Extended Diploma or SQCF Level 6) may be accepted as an alternative

Minimum offer level

A level: A*A*A
IB: 41-42 points, with 776 at Higher Level
STEP: all Colleges require at least grade 1 in two STEP papers (STEP 2 and 3)
Other qualifications: Check which other qualifications we accept.

Studying X subject at University

UCAS Subject Page

UCAS/mathematics-and-statistics

UCAS/physics-and-astronomy

Cambridge Course Page

https://www.undergraduate.study.cam.ac.uk/courses/mathematics-ba-hons-mmath

https://www.undergraduate.study.cam.ac.uk/courses/natural-sciences-ba-hons-msci

myheplus.com Subject Page

https://myheplus.com/subject/mathematics

https://myheplus.com/subject/physics

Thank you