Topic 5: Geometry

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Introduction

This topic will cover some basic concepts in (non-)Euclidean and Riemannian geometry. We start by refreshing some concepts from Euclidean geometry, focussing on the Euclidean metric and the properties of straight lines. Afterwards, we will see how the intuition that we have built up in Euclidean geometry is challenged when we consider geometry on a sphere.

We then move on to a very brief introduction of Riemannian geometry, which is the study of manifolds equipped with a Riemannian metric. The details of this topic would lead us too far, but we will try to get some feeling for the relevant concepts, focussing on the metric and geodesics.

Finally, we will introduce Lorentzian geometry and its applications in Special and General Relativity. We will see how the geometry of spacetime is described by a Lorentzian metric, and how this metric can be used to describe the motion of particles and light rays in spacetime. Of course, a detailed mathematical treatment would take us too far, but the aim is to give a flavour of the concepts involved.

These notes are compact, and are further supported by a slide presentation, that can also be found on the webpage. The slide presentation contains more figures to support these notes.

Lecture 1: Euclidean geometry and spherical geometry

This section summarizes the key concepts in Euclidean geometry and metric spaces.

1.1 Euclidean geometry

We start by defining the Euclidean metric, which is the distance function that we are used to from high school.

Definition 1 (Euclidean metric). *The Euclidean metric on $\mathbb{R}^n$ is defined by*

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \tag{1}$$

for $x, y \in \mathbb{R}^n$.

For $n = 2$, this is the familiar distance formula in the plane:

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\,,$$

and can also be interpreted as Pythagoras' theorem:

$$c^2 = a^2 + b^2\,.$$

Keeping the next sections in mind, we will often denote (1) as

$$\mathrm{d}s^2 = \sum_{i=1}^{n} \mathrm{d}x_i^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \ldots + \mathrm{d}x_n^2\,. \tag{2}$$

Again, for $n = 2$, this gives

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2\,.$$

1.2 Straight lines

We all have an intuitive understanding of what a straight line is. In Euclidean geometry, a straight line is the shortest path between two points. It also has the following properties

- Through any 2 different points goes exactly one straight line.
- Two parallel lines either are the same, or never intersect.
- Two different straight lines intersect at most once.

As we will see, these properties are not necessarily true in other geometries.

1.3 Geometry on a sphere

On a sphere, the Euclidean metric does not make sense. Indeed, we can only move along the surface of the sphere, and not through it: therefore, the distance formula (1) does not make sense. Additionally, "straight lines" no longer exist on the sphere: any line will be curved.

Instead, we need to define a new metric that will give us the correct distances between two points on a sphere. We can label the points on a sphere using two angles, θ and ϕ. The metric on a sphere, for two points that are "close together", is given by

Definition 2 (Spherical metric)**.** *The spherical metric on the sphere with radius R is given by*

$$ds^2 = R^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)\,. \tag{3}$$

Compare this expression to the Euclidean metric (1). It is still of the same form, where the distance depends on the coordinate differences squared. However, this time there are factors multiplying these differences, which even depend on the coordinates themselves. This will be a general feature of Riemannian metrics, as we will see in Lecture 2.

Importantly, the metric on a sphere, as defined above, can be used to calculate the distance between points that are "infinitesimally close". To find the distance between two points that are further apart, we need to integrate the metric along the path between the two points. This gives us a formula for the distance between two points on a sphere:

$$d\left((\theta_1, \phi_1), (\theta_2, \phi_2)\right) = \int_{(\theta_1,\phi_1)}^{(\theta_2,\phi_2)} \mathrm{d}s = \int_{(\theta_1,\phi_1)}^{(\theta_2,\phi_2)} R\sqrt{\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2}\,. \tag{4}$$

It is not immediately clear how one would actually use this formula to calculate the distance, but the idea should be clear: we calculate the distance between two distant points by summing (integrating) over the distances between infinitesimally close points. The accompanying slide presentation contains an example that illustrates how the distance between two points differs when we use the Euclidean metric and the spherical metric.

As mentioned before, we no longer have straight lines on a sphere. Recall, however, the characterization of a straight line as being the shortest path between two points. On a sphere, we can still look for the shortest path between two points. This is called a **geodesic**. It turns out that geodesics on a sphere are great circles.

Definition 3 (Great circle)**.** *A great circle is the intersection of a sphere with a plane that passes through the centre of the sphere.*

1.3.1 *Optional*: The geodesic equation on a sphere

The following is very advanced material, and I do not attempt to explain everything in detail. Below is a high-level brief summary of some concepts, to try and give a bit of background to the geodesic equation for the sphere.

To understand the geodesic equation, we briefly need to understand index notation. In index notation, we write the metric as a matrix g_{ij}, where i, j are indices that run from 1 to the dimension of the manifold. The line element is then given by

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j \,. \tag{5}$$

In this expression, we sum over all repeated indices, which is called the Einstein summation convention. The x^i are the coordinates of the manifold; $x^1 = \theta$ and $x^2 = \phi$ in the case of the sphere. If we compare this to the spherical metric (3), we see that g_{ij} is given by

$$g_{ij} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix} \,. \tag{6}$$

The geodesic equation is a differential equation that describes the geodesics on a manifold. It is expressed in terms of the metric g_{ij} and the Christoffel symbols Γ^i_{jk}, which are related to the metric by[1]

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}\right) \,. \tag{7}$$

In this expression, we have also introduced the inverse metric g^{ij}, which is the inverse of the matrix g_{ij}. Furthermore, the ∂_i are the partial derivatives with respect to the coordinates x^i, i.e.

$$\partial_i = \frac{\partial}{\partial x^i} \,. \tag{8}$$

This, in principle, gives us all the necessary ingredients to write down the geodesic equation. We first have to calculate the Christoffel symbols. It can be shown that the non-zero Christoffel symbols for the sphere are

$$\begin{aligned} \Gamma^\theta_{\phi\phi} &= -\sin\theta\,\cos\theta \\ \Gamma^\phi_{\theta\phi} &= \Gamma^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta} \,. \end{aligned}$$

We are now finally ready to understand the geodesic equation

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} = 0 \,. \tag{9}$$

[1] From the definition here it follows that the Christoffel symbols are symmetric in the lower indices, i.e. $\Gamma^i_{jk} = \Gamma^i_{kj}$.

Note that there is an equation for every coordinate x^i, as that index is not summed over. In this equation, λ is an affine parameter that parametrizes the curve. For the sphere, the geodesic equations are

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\lambda^2} - \sin\theta\,\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 = 0\,,$$
$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\lambda^2} + 2\frac{\cos\theta}{\sin\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = 0\,.$$

One can attempt to solve these in generality, but it is not straightforward. Therefore, a trick is used: since the sphere has a high degree of symmetry, we can exploit this symmetry to find the geodesics. As the geodesic equations are second order differential equations, they require two initial conditions to be solved, a position (θ_0, ϕ_0) and a velocity $(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\big|_0, \frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\big|_0)$. Due to the high degree of symmetry, the sphere can always be rotated such that the initial position has $\theta_0 = \frac{\pi}{2}$, as well as $\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\big|_0 = 0$. As $\theta_0 = \frac{\pi}{2}$, we have $\cos\theta_0 = 0$. Therefore, the initial geodesic equations simplify to

$$\left.\frac{\mathrm{d}^2\theta}{\mathrm{d}\lambda^2}\right|_0 = 0\,,$$
$$\left.\frac{\mathrm{d}^2\phi}{\mathrm{d}\lambda^2}\right|_0 = 0\,.$$

This means that $\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}$ does not change, and stays zero, meaning that θ remains equal to $\frac{\pi}{2}$. Therefore, the initial equations retain their form. The solution to these simultaneous equations is the great circle that is the equator, traversed at constant angular velocity:

$$\gamma(\lambda) = \left(\frac{\pi}{2}, \phi_0 + \lambda\left.\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right|_0\right)\,, \tag{10}$$

where ϕ_0 is the initial azimuthal angle and $\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\big|_0$ is the constant initial angular velocity.

Lecture 2: Riemannian geometry

The example of spherical geometry showed that our intuition can be challenged when we look beyond Euclidean geometry. Introducing a new metric, i.e. a new "way to measure distances", changes our geometry.

Riemann presented a way to generalize Euclidean geometry to arbitrary manifolds (roughly speaking, a space in which we want to do geometry that satisfies some properties), using fairly general metrics. Examples of manifolds can include a sphere, a torus, higher dimensional shapes ... A precise definition of a manifold is beyond the scope of this course, however.

Furthermore, he allowed the metric to take a much more general form:

$$\mathrm{d}s^2 = g_{11}(\mathbf{x})\mathrm{d}x_1^2 + g_{22}(\mathbf{x})\mathrm{d}x_2^2 + g_{12}(\mathbf{x})\mathrm{d}x_1\mathrm{d}x_2 + \dots \; . \tag{11}$$

Note that the **metric components** $g_{11}, g_{22} \dots$ can depend on the coordinates $\mathbf{x}$, much like the metric on the sphere depends on θ in the factor $\sin^2\theta$. This means that "distances" can vary from point to point, in a certain sense. Secondly, the metric can also depend on *cross-terms* of the form $\mathrm{d}x_1\mathrm{d}x_2$.

An important property of Riemannian geometry is that distances are still always positive; when integrating along a curve between two points, the length of the curve is always positive. We will see that this is no longer true once we move to Lorentzian geometry.

2.1 Poincaré half-plane model

As an example of Riemannian geometry, we consider the Poincaré half-plane model. The manifold under consideration is the upper half of the Euclidean plane, excluding the x-axis itself. More precisely, we consider

$$\mathbb{H} = \left\{(x, y) \in \mathbb{R}^2 \mid y > 0\right\} \; . \tag{12}$$

The metric on this manifold is given by

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} \; . \tag{13}$$

This metric is a satisfactory Riemannian metric and well-defined on the Poincaré half-plane. So, this metric suggests that "distances become larger as we move towards the x-axis".

A question one could ask now is what the geodesics on this manifold look like (remember that geodesics are the generalization of straight lines, minimizing the distance between two points). It turns out that the geodesics on the Poincaré half-plane are

1. straight vertical lines,
2. or arcs of circles with centre on the x-axis.

These geodesics are illustrated in the slides.

2.1.1 *Optional*: The geodesic equation on the Poincaré half-plane

The geodesic equation on the Poincaré half-plane can be derived similarly as for the sphere (Sec. 1.3.1). The non-zero Christoffel symbols for the Poincaré half-plane are

$$\begin{aligned} \Gamma^y_{yy} &= -\frac{1}{y} \, , \\ \Gamma^x_{xy} = \Gamma^x_{yx} &= -\frac{1}{y} \\ \Gamma^y_{xx} &= \frac{1}{y} \, . \end{aligned}$$

Therefore, we find the geodesic equations

$$\begin{aligned} \frac{\mathrm{d}^2 x}{\mathrm{d}\lambda^2} - \frac{2}{y}\frac{\mathrm{d}x}{\mathrm{d}\lambda}\frac{\mathrm{d}y}{\mathrm{d}\lambda} &= 0\,, \\ \frac{\mathrm{d}^2 y}{\mathrm{d}\lambda^2} - \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)^2 + \frac{1}{y}\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2 &= 0\,. \end{aligned}$$

Again, we need two initial conditions, the initial position (x_0, y_0) and the initial velocity $(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\big|_0, \frac{\mathrm{d}y}{\mathrm{d}\lambda}\big|_0)$. The first case we consider is when the initial velocity is zero along the x-axis, i.e. $\frac{\mathrm{d}x}{\mathrm{d}\lambda}\big|_0 = 0$. This means that the geodesic equations simplify to

$$\begin{aligned} \left.\frac{\mathrm{d}^2 x}{\mathrm{d}\lambda^2}\right|_0 &= 0\,, \\ \left.\frac{\mathrm{d}^2 y}{\mathrm{d}\lambda^2}\right|_0 - \frac{1}{y_0}\left(\left.\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right|_0\right)^2 &= 0\,. \end{aligned}$$

Again, we see that the velocity along the x-axis does not change, and stays zero. The solution is a vertical line that gets traversed at varying speed.

The solution where $\frac{\mathrm{d}x}{\mathrm{d}\lambda}\big|_0 \neq 0$ is a fair bit more complicated, and we do not discuss it here.

Lecture 3: Lorentzian geometry and relativity

For this section, I will refer to the slides, with the addition of the following clarification. A metric for which distances no longer need to be positive is no longer Riemannian. It is complicated to define a Lorentzian metric without some more background, but roughly speaking, a Lorentzian metric is a metric that has one "timelike" direction, meaning that there is one coordinate along which distances are negative. Other more mathematical formulations of this statement would be "the metric has signature $(-, +, +, +, \dots)$" or "the metric (as a matrix) has exactly one negative eigenvalue".